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NON-LINEAR REFLECTION OF A PLANE WAVE IN AN ELASTIC MEDIUM†

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Exact formulae are derived for the reflected and refracted waves which occur for the inclined incidence of a plane horizontally polarized transverse wave of arbitrary profile on a horizontal interface between two elastic half-spaces experiencing non-linear friction when they move with respect to one another. A smooth function of general form is adopted as the friction function, which depends on the difference between the horizontal velocities of the elements of the boundaries of the half-spaces considered. It is shown that if the friction function depends non-monotonically on the relative velocity of displacement of the sides of a slit, then even when the profile of the incident wave is smooth, the reflected and refracted waves may contain strong discontinuities.

A somewhat similar problem was solved in [1, 2] for the case of piecewise-constant friction with possible slippage; particular attention was devoted to the case when the incident wave is a rectangular pulse.

Suppose the space *xyz* consists of two half-spaces: $z > 0$ (medium 1 with shear modulus μ_1 and density ρ_1) and $z < 0$ (medium 2 with shear modulus μ_2 and density ρ_2).

We will consider only plane horizontally polarized transverse waves (*SH*-waves). Without loss of generality we will assume that the vectors of the normals to the wave fronts of all the propagating waves lie in the *xz* plane; consequently, only the *y*-component of the displacement is non-zero.

We recall the following well-known relation between the stresses and displacements $\mathbf{u} = (0, u, 0)$ in a plane *SH*-wave propagating in a linearly elastic medium with shear modulus μ

$$(\sigma_{zx}, \sigma_{yz}, \sigma_{zz}) = (0, \mu\partial u/\partial z, 0) \tag{1}$$

We will assume that the half-spaces $z > 0$ and $z < 0$ experience non-linear friction with horizontal slippage with respect to one another. In other words, we will assume that the friction forces $\mathbf{F}_{\text{fr}}^1$, acting on unit area of the boundary surface of the half-space $z > 0$ has the form

$$\mathbf{F}_{\text{fr}}^1 = (0, -F(\partial(u^+ - u^-)/\partial t), 0); \; u^{\pm} = u|_{z=0\pm} \tag{2}$$

were F is an arbitrary smooth monotonically increasing function such that $F(0) = 0$. We will denote the friction force on unit area of the boundary surface of the half-space $z < 0$ by $\mathbf{F}_{\text{fr}}^2$. By Newton's third law we have

$$\mathbf{F}_{\text{fr}}^2 = -\mathbf{F}_{\text{fr}}^1 \tag{3}$$

It is clear that the condition for the sum of the forces acting on an infinitely thin element of medium 1 adjoining the boundary $z = 0$ to be zero has the form $\sigma_{yz} + (\mathbf{F}_{\text{fr}}^1)_y = 0$, whence, by virtue of (1) and (2), we have

$$\mu_1 \partial u/\partial z|_{z=0+} = F(\partial(u^+ - u^-)/\partial t) \tag{4}$$

A similar condition for medium 2 is

$$\mu_2 \partial u/\partial z|_{z=0-} = F(\partial(u^+ - u^-)/\partial t) \tag{5}$$

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Finally, we write the incident *SH*-wave of displacements $\mathbf{u}^{in} = (0, u^{in}, 0)$ in the form

$$u^{in} = f\left(t - \frac{\sin j}{\beta_1}x + \frac{\cos j}{\beta_1}z\right), \ z > 0 \tag{6}$$

Here $\beta_s = (\mu_s/\rho_s)^{1/2}$ is the velocity of shear waves in medium s ($s = 1, 2$), j is the acute angle between the direction of propagation of the wave and the z axis, and $f(\xi)$ is an arbitrary smooth function equal to zero when $\xi < 0$. The problem is to determine the reflected and refracted waves from (4)–(6). To fix our ideas we will confine ourselves to the case when

$$\beta_1 > \beta_2 \tag{7}$$

(Condition (7) includes the possibility of total internal reflection in (4)–(6).)

It is clear that in this problem only the y-component in the reflected and refracted waves will be non-zero, i.e.

$$\mathbf{u}^{ref} = (0, u^{ref}, 0), \ \mathbf{u}^{tr} = (0, u^{tr}, 0)$$

We will seek u^{ref} and u^{tr} in the form

$$u^{ref} = \varphi\left(t - \frac{\sin j}{\beta_1}x - \frac{\cos j}{\beta_1}z\right), \ z > 0 \tag{8}$$

$$u^{tr} = \psi\left(t - \frac{\sin k}{\beta_2}x + \frac{\cos k}{\beta_2}z\right), \ z < 0 \tag{9}$$

where k is the acute angle between the direction of propagation of the wave and the z axis, not known in advance.

In order to determine the unknown functions φ and ψ we used boundary conditions (4) and (5) where, when $z = 0+$ we must put $u = u^{in} + u^{ref}$, and when $z = 0$ we must put $u = u^{tr}$.

Thus, from (4) we have

$$[f'(\xi) - \varphi'(\xi)]\mu_1\beta_1^{-1}\cos j = F(f'(\xi) + \varphi'(\xi) - \psi'(\eta)) \tag{10}$$

$$\xi = t - x\beta_1^{-1}\sin j, \ \eta = t - x\beta_2^{-1}\sin k$$

and from (5) we have the similar relation

$$\psi'(\eta)\mu_2\beta_2^{-1}\cos k = F(f'(\xi) + \varphi'(\xi) - \psi'(\eta)) \tag{11}$$

It is obvious that for (10) and (11) to be satisfied identically, for all t and x we must have

$$\beta_1^{-1}\sin j = \beta_2^{-1}\sin k \tag{12}$$

By virtue of (7), equality (12) defines a real value of the angle k, $0 \leqslant k < \pi/2$ for all j, $0 \leqslant \ < \pi/2$.

Now, taking (12) into account, we can write (10) and (11) in the form of the following system

$$\begin{aligned} &[f'(\xi) - \varphi'(\xi)]\mu_1\beta_1^{-1}\cos j = F(f'(\xi) + \varphi'(\xi) - \psi'(\xi)) \\ &\psi'(\xi)\mu_2\beta_2^{-1}\cos k = F(f'(\xi) + \varphi'(\xi) - \psi'(\xi)) \end{aligned} \tag{13}$$

which contains two unknown functions φ' and ψ'. Note, however, that system (13) can be reduced to a single functional equation in a single unknown function. In fact, subtracting the first equation of (13) from the second and substituting the result into the first equation of (13) we obtain an equation for φ'

$$\mu_1\beta_1^{-1}\cos j f'(\xi) = \mu_1\beta_1^{-1}\cos j\varphi'(\xi) + F(f'(\xi)[1-\kappa] + \varphi'(\xi)[1+\kappa]), \tag{14}$$

$$\kappa = \mu_1\beta_2 \cos j / (\mu_2\beta_1 \cos k)$$

Since $1 + \kappa > 0$, $\mu_1\beta_1^{-1}\cos j > 0$, the right-hand side of (14) is a strictly increasing continuous function of φ' (for any continuous monotonically increasing function F). Consequently, Eq. (14) can be solved uniquely for φ', where $\varphi'(\xi)$ is a continuous function (simultaneously with $f(\xi)$). We will denote the corresponding function by

$$\varphi'(\xi) = G(f'(\xi)) \tag{15}$$

Obviously, we must put

$$\varphi(\xi) = \int_0^\xi G(f'(\xi))d\xi \tag{16}$$

since in the region where the incident wave has not travelled the displacement can be assumed to be zero. We then have from (16) and (13)

$$\psi(\xi) = \left[f(\xi) - \int_0^\xi G(f'(\xi))d\xi\right]\kappa \tag{17}$$

The function $\psi(\xi)$ defined by (15) is obviously also continuous. Formulae (16) and (17) are the required solution of the problem in question.

Note. Suppose the function $f(\xi)$ is non-zero only when $0 < \xi < A; A > 0$. Then, if the function F is not identical with a linear function in any part, we have, generally speaking

$$\varphi(A) = \int_0^A G(f'(\xi))d\xi \neq 0 \tag{18}$$

Hence, on the ray $[A, \infty)$ we have $\varphi(\xi) \equiv \varphi(A)$ and $\psi(\xi) \equiv \psi(A) = -\kappa\varphi(A)$ (see (17)). Hence, the quantities $\varphi(A)$ and $\psi(A)$ are residual constant displacements of the half-spaces $z > 0$ and $z < 0$, which occur after the incident wave has passed through.

The case when the function F is non-monotonic is also of interest. We will introduce the notation $g(\xi) = f(\xi)[1 - \kappa] + \varphi'(\xi)[1 + \kappa]$. Then (14) can be rewritten in the form

$$2\nu f'(\xi) = \nu g(\xi) + F(g(\xi)) \tag{19}$$

$$(\nu = \mu_1\mu_2 \cos j \cos k / (\mu_1\beta_2 \cos j + \mu_2\beta_1 \cos k))$$

The right-hand side of (19) can obviously be both a monotonic and a non-monotonic function of $g(\xi)$ depending on the behaviour of the function F. If the function F is such that the right-hand side of (19) is monotonic (as a function of $g(\xi)$), then it is obvious that all the conclusions reached above hold.

Suppose now that the right-hand side of (19) is a non-monotonic function if $g(\xi)$. Then if the range of values of $f(\xi)$ is sufficiently large (i.e. the profile of the incident wave is sufficiently sharp), then for certain ξ, Eq. (19) will have a single solution. In addition, it is geometrically obvious that in this situation every unique branch $\tilde{g}(\xi)$ of the solution of Eq. (19) is inevitably discontinuous. The functions $\varphi'(\xi)$ and $\psi'(\xi)$ will thereby also be discontinuous, i.e. the reflected and refracted waves of displacements imply discontinuities of the first derivatives.

Hence, in the case of a non-monotonic friction function, non-linear reflection and refraction at the interface between two linear media may be the mechanism by which discontinuities arise in wave problems.

Note that the problem of strong discontinuities in the friction function does not necessarily imply the formation of strong discontinuities in the reflected and refracted waves. For example, if $F(g) = k \operatorname{sign} g$, $k > 0$, then for a continuous function $f(\xi)$ there is obviously a continuous solution $g(\xi)$ of (19) since the right-hand side of (19) is a monotonic function of g. Hence, the mechanism that gives rise to discontinuities of the reflected and refracted waves is, in fact, the non-monotonic form of the friction function and not the presence of strong discontinuities in it. It is also easy to see that the discontinuities that arise due to the non-monotonic form of the friction function are stable with respect to small changes of this function and also with respect to small changes in the profile of the incident wave. Note, finally, that the non-monotonic form of the friction function leads, under certain conditions, to the occurrence of self-excited oscillations in certain related problems in geophysics [3].

The method proposed in this paper can be extended to the case of horizontally stratified media, when each of the strata is uniform, isotropic and linearly elastic and experiences non-linear friction when there is horizontal displacement with respect to a neighbouring layer.

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